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LETTER TO THE EDITOR

Basic conservation laws in the electromagnetic theory of cyclotron radiation: further analysis

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Abstract. The conflict of basic conservation laws in cyclotron radiation is considered in more general terms, taking into account relativistic effects of the electron (i.e. synchrotron radiation). We also investigate effects due to the most important approximation in cyclotron theory, viz the omission of radiation back reaction. Our conclusions are (i) the disagreement is of a magnitude considerably larger than any errors introduced by the approximation; (ii) the 'degree of conflict' attains its maximum in relativistic velocities, when the energy loss to radiation can approach the total energy of the electron.

In an earlier letter (Lieu *et al* 1983, hereafter referred to as I) it was shown that, during cyclotron radiation from a non-relativistic electron, an incompatibility exists among the conservation laws of energy, angular momentum, and linear momentum. In this work we generalise the arguments to include the relativistic domain (sometimes called synchrotron radiation). We also examine the effects of an important approximation used in classical electromagnetic theory, in an attempt to search for a method of resolution.

For a relativistic electron in a uniform and constant magnetic field (and ignoring motion along the field, z-direction) the angular frequency of circular motion is given by

$$\omega_0 = \omega_c/\gamma \tag{1}$$

where γ is the Lorentz factor, and $\omega_c = eB/m$ is the angular frequency in the limit of zero speed. The radius of gyration is given by

$$r_1 = v/\omega_0 = \gamma v/\omega_c \tag{2}$$

where v is the transverse velocity. The transverse momentum has magnitude $p = m\gamma v = m\omega_c r_1$. Use can then be made of the relativistic energy–momentum relationship to deduce the electron energy as

$$E_e = (1 + \omega_c^2 r_1^2/c^2)^{1/2} mc^2. \tag{3}$$

During cyclotron radiation, there is a continual decrease in electron energy and orbital radius:

$$dE_e/dt = [(m\omega_c c)^2/2E_e]\, dr_1^2/dt \tag{4}$$

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where, by energy conservation,

$$dE_e/dt = -dE_\gamma/dt = \eta \tag{5}$$

and η, the rate of energy loss to radiation, is related to the $j(\omega,\theta)$ in the appendix of I by

$$\eta = \sum_{m=1}^{\infty} \int_0^{\pi} j(\omega, \theta) 2\pi \sin\theta \, d\theta. \tag{6}$$

The summation and integration can be performed to yield (Bekefi 1966)

$$\eta = (e^2\omega_c^2/6\pi\varepsilon_0 c^3)\gamma^2 v^2. \tag{7}$$

Turning now to z-angular momentum, the definition for a relativistic electron is (equation (31), Johnson and Lippmann 1949)

$$L_z^e = (2eBc^2)^{-1}(E_e^2 - m^2c^4) - \tfrac{1}{2}(m\omega_c r_0^2)$$

where r_0 is the radial position of the guiding centre. This, together with (3), gives

$$L_z^e = \tfrac{1}{2}m\omega_c(r_1^2 - r_0^2) \tag{8}$$

which is identical to the non-relativistic definition (equation (19) of Johnson and Lippmann 1949). Angular momentum conservation during radiation can be written as

$$dL_z^e/dt = -dL_z^\gamma/dt \tag{9}$$

where, as explained in I,

$$dL_z^\gamma/dt = (1/\omega_0)\, dE_\gamma/dt \tag{10}$$

(note that (10) differs from (5) of I by the replacement $\omega_c \to \omega_0$, the relativistic cyclotron frequency). Use of (5), (9) and (10) leads to

$$dL_z^e/dt = (1/\omega_0)\, dE_e/dt$$

which, together with (1), (4) and (8) gives

$$dr_0^2/dt = 0 \tag{11}$$

and the conclusion, viz., that there is no net radial drift of the guiding centre, is the same as that obtained in the non-relativistic limit.

Linear momentum conservation, on the other hand, does not provide agreement with the above. In fact, all the relations in I on linear momentum can be applied to the present relativistic context without modification. The final results were

$$dx_0/dt = (1/m\omega_c)\, dP_y/dt, \qquad dy_0/dt = -(1/m\omega_c)\, dP_x/dt \tag{12a}$$

and

$$dr_0^2/dt = (m\omega_c)^{-2}\, dP^2/dt. \tag{12b}$$

The time-averaged values of $\dot{P}_y$ and $\dot{P}_x$, whence $\dot{x}_0$ and $\dot{y}_0$, are zero. The time average of $\dot{P}^2$ is in general finite, so that (11) and (12b) reveals the presence of conflict in the conservation lows.

Close examination of (12) reveals that the guiding centre is 'spiralling' outwards about its initial position. The frequency of the spiral is determined by the frequency of the radiation. In the non-relativistic limit, $v/c \ll 1$, only the fundamental frequency

$\omega=\omega_0=\omega_c$ is emitted. In the relativistic limit, $v/c\simeq 1$ emission is predominantly in the high harmonics, $\omega\simeq\gamma^3\omega_0$ (Bekefi 1966). It is evident that the guiding centre motion always occurs in characteristic time scales $\lesssim\omega_0^{-1}$. Time average over one cyclotron period $2\pi/\omega_0$ is thus seen to be sufficient for any quantity.

It is also useful to examine the quantity $\mathrm{d}P^2/\mathrm{d}t=2P\,\mathrm{d}P/\mathrm{d}t$ in (12*b*). The appendix of I gives the time-averaged value of $\mathrm{d}P/\mathrm{d}t$:

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\sum_{m=1}^{\infty}\int_0^{\pi} j(\omega,\theta)(\sin\theta/c)2\pi\sin\theta\,\mathrm{d}\theta \tag{13}$$

and this is constant for times $\Delta t\sim$ (a few) $\times 2\pi/\omega_0$. This means P, and hence $\mathrm{d}P^2/\mathrm{d}t$ increases linearly with time during the interval Δt. Such results are useful to subsequent arguments.

It is now important to investigate whether the paradox owes its origin to an inexact theory. Within the context of classical electromagnetism, the only significant approximation made is the omission of radiation back reaction. More specifically, there exists an error in the basic emission coefficient η (equation (6)) because it is computed on the assumption that the amount of radiative energy loss per cyclotron period is small. This leads to uncertainties in the dynamical variables computed.

The basic unit of uncertainty in this problem is $\delta\eta/\eta$, the fractional error in the emission coefficient. From (2) and (7) we deduce

$$\delta\eta/\eta=2\,\delta r_1/r_1.$$

We may now use (3) to rewrite this as

$$\delta\eta/\eta=[2\gamma/(\gamma^2-1)]\varepsilon/mc^2 \tag{14}$$

where ε is the (finite) energy loss per cycle:

$$\varepsilon=\gamma(\gamma^2-1)e^2\omega_c/3\varepsilon_0 c. \tag{15}$$

We also note in passing that, in the non-relativistic limit $\gamma\simeq 1$, (14) may be rewritten as

$$\delta\eta/\eta=\varepsilon/T \tag{16}$$

where $T=\frac{1}{2}mv^2$ is the kinetic energy of the electron.

Combining (14) and (15) we have

$$\delta\eta/\eta=\gamma^2 2e^2\omega_c/3\varepsilon_0 mc^3=1.38\times10^{-11}\gamma^2 B \tag{17}$$

where B is the magnetic field in Tesla. In a 2 T field, an eV electron has $\delta\eta/\eta\sim10^{-11}$, and a GeV electron has $\delta\eta/\eta\sim10^{-4}$.

Having established the framework we can proceed to calculate the fractional error in $\dot{r}_0^2$ if linear momentum conservation (12*b*) is obeyed. First we attempt to express $\mathrm{d}P/\mathrm{d}t$ in (13) in terms of η in (6) and (7). Exact relations can be obtained in the non-relativistic and ultra-relativistic limits. In the former case we can use the expression of $j(\omega,\theta)$ as given in the appendix of I, together with the limiting behaviour of Bessel functions, viz $J_n(z)\sim z^n/(2^n n!)$ for $z\ll 1$. Retaining only the lowest-order terms, it is not difficult to show that

$$\mathrm{d}P/\mathrm{d}t=\tfrac{15}{64}\pi c^{-1}\,\mathrm{d}E_\gamma/\mathrm{d}t=\tfrac{15}{64}\pi\eta/c.$$

Integrating, subject to the boundary condition that there is no radiation at $t<0$, we get

$$P=\tfrac{15}{64}\pi E_\gamma/c.$$

In the ultra-relativistic limit $j(\omega, \theta)$ is sharply peaked at $\theta \simeq \frac{1}{2}\pi$, $\sin\theta \simeq 1$ (Bekefi 1966). Hence a comparison of (6) and (13) reveals that

$$\mathrm{d}P/\mathrm{d}t = c^{-1}\,\mathrm{d}E_\gamma/\mathrm{d}t=\eta/c.$$

For intermediate energies, the relations approximately hold, viz

$$\mathrm{d}P/\mathrm{d}t \simeq c^{-1}\,\mathrm{d}E_\gamma/\mathrm{d}t=\eta/c, \qquad P\simeq E_\gamma/c. \tag{18}$$

Substituting (18) into (12*b*) we obtain

$$\mathrm{d}\dot{r}_0^2/\mathrm{d}t \simeq [2/(m\omega_c c)^2]E_\gamma\eta \tag{19}$$

so that the fractional error in $\mathrm{d}r_0/\mathrm{d}t$ is

$$\delta\dot{r}_0^2/\dot{r}_0^2 \simeq \delta\eta/\eta + \delta E_\gamma/E_\gamma \simeq 2\delta\eta/\eta$$

where the final expression is obtained from $E_\gamma=\eta t$ (a more exact treatment, which takes into account the change in η per cycle, leads to the same result). From the value of $\delta\eta/\eta$ given in (17) we can see that $\delta\dot{r}_0^2/\dot{r}_0^2 \ll 1$. This means $\dot{r}_0^2$ is a genuine quantity whch is not removed by the presence of radiation back reaction.

The 'degree of conflict' among the conservation laws may be defined as the ratio $R=\dot{r}_0^2/\dot{r}_1^2$ where $\dot{r}_0^2$ is given by (19), and $\dot{r}_1^2$ by (4). Before we do so, however, it is necessary to compute the ratio $S=\delta\dot{r}_1^2/\dot{r}_0^2$. The conflict is real *only* if we can establish that $S \ll 1$, i.e., the finiteness of $\dot{r}_0^2$ from linear momentum conservation does upset the other conservation laws by an amount large compared with their limits of uncertainty. From (4) and (19) we deduce that

$$S=\delta\dot{r}_1^2/\dot{r}_0^2=(E_e/E_\gamma)\delta\eta/\eta+\delta E_e/E_\gamma \simeq (1+E_e/E_\gamma)\delta\eta/\eta.$$

Since $\delta\eta/\eta \ll 1$, it is sufficient to show that $(E_e/E_\gamma)(\delta\eta/\eta) \ll 1$. In the relativistic limit this is clearly so, since the total energy output E_γ can be comparable to the electron energy, i.e. $E_e/E_\gamma \to 1$. In the non-relativistic limit, we make use of (16) to show that

$$(E_e/E_\gamma)\delta\eta/\eta=(mc^2/n\varepsilon)\varepsilon/T=mc^2/nT$$

where n is the number of cycles and ε (as defined before) is the energy loss per cycle. From (17) we realise that, for $B\sim 1$ T and $\gamma \simeq 1$, the system can remain quite unchanged after 10^9–10^{10} cycles. For $n \sim 10^9$ and $T \sim 1$ eV we have $mc^2/nT \sim 10^{-3}$. The fact that S can be $\ll 1$ is thus established for all electron energies.

We may now proceed with somewhat more confidence on the 'degree of conflict', R, defined earlier. From (4) and (19) we deduce that

$$R=\dot{r}_0^2/\dot{r}_1^2=E_\gamma/E_e=\text{energy output/electron total energy}.$$

The 'degree of conflict' is more important in relativistic energies. For reasons given earlier, R can approach 100% for a synchrotron.

In summary, the problem addressed in I is shown to persist, with a greater degree of severity, in relativistic energies. Moreover, it is shown to be unrelated to the approximation used in cyclotron radiation theory. It appears extremely plausible that a similar incompatiblity exists, in a more fundamental level, between kinematic conservation laws and the underlying axioms of classical electromagnetism. It is also unlikely that phenomena outside the scope of electrodynamics can be responsible. Effects such as quantum uncertainties, electron spin, finite electron size, and weak interaction are too small in the energies of interest here.

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References

Bekefi G 1966 *Radiation Processes in Plasmas* (New York: Wiley)
Johnson M H and Lippmann B A 1949 *Phys. Rev.* **76** 828
Lieu R, Leahy D A and Evans A J 1983 *J. Phys. A: Math. Gen.* L669